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A Note on the LFM Time Integration

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1. Introduction

The occurrence of unrealistic 500 mb vorticity "wave trains" and associated disturbances in the sea level pressure field had been noted on many occasions. A series of numerical experiments was carried out in an effort to overcome these model deficiencies. A fair degree of success has been achieved by modification of the smoothing operation performed as a part of the time integration of the LFM. This note will present the elements of the change introduced and a simple numerical analysis of the revised method.

2. Numerical Method

The time integration technique used in the LFM model may be simply analyzed by treating the idealized case of a pure gravity wave. Let ϕ denote a geopotential (of a free surface) and u denote the fluid speed. The equations are

$$\frac{\partial u}{\partial t} = - \frac{\partial \phi}{\partial x}$$

$$\frac{\partial \phi}{\partial t} = - c^2 \frac{\partial u}{\partial x} \qquad (1)$$

in which c is the phase speed of the wave.

The numerical method of time integration may be expressed by the equations

$$\frac{u^{n+1} - \overline{u^{n-1}}}{2\Delta t} = - i \hat{k} \phi^n$$

$$\frac{\phi^{n+1} - \overline{\phi^{n-1}}}{2\Delta t} = - c^2 i \hat{k} u^n \qquad (2)$$

The superscript n denotes the time step counter; $\hat{k}$ is the horizontal wave number incorporating spatial truncation error; Δt is the time step interval, and $i = \sqrt{-1}$. It has been assumed that

$$\begin{matrix} u \\ \phi \end{matrix} \sim \begin{matrix} u^n \\ \phi^n \end{matrix} e^{ikx} \tag{3}$$

and we may note that

$$\hat{k} = + \left(\frac{\sin k\Delta x}{k\Delta x} \right) k \ . \tag{4}$$

The bar overriding the n-1 values denotes a spatial smoothing of those fields. The smoother now used in the LFM is a 25-point operator at all points two or more grid intervals removed from the boundary. At points only one interval from the boundary, a nine-point operator is used.

For simplicity, we consider only the one-dimensional (five point) form of the operator that is applied at the bulk of the interior grid points. The operator involves only one parameter, β, and has the form

$$\overline{u(x)} = - \beta^2 u(x-2\Delta x) + 4\beta^2 u(x-\Delta x) + (1-6\beta^2) u(x)$$
$$+ 4\beta^2 u(x+\Delta x) - \beta^2 u(x+2\Delta x) \tag{5}$$

When the operator defined by (5) is passed over a wave, of the form given in (3), one obtains

$$\overline{u} = R(k)u \tag{6}$$

with $$R(k) = 1 - 2\beta^2[3 - 4 \cos k\Delta x + \cos 2\, k\Delta x] \tag{7}$$

or $$R(k) = 1 - [2\beta(1 - \cos k\Delta x)]^2 \tag{8}$$

A stability analysis of the system of equations (2) can be carried out by assuming that

$$\begin{matrix} u^n \\ \phi^n \end{matrix} = \begin{matrix} U \\ P \end{matrix} \zeta^n \tag{9}$$

Provided that $|\zeta|$ is less than unity, the numerical solution is considered to be linearly stable.

Substitution of (9) into (2) yields a pair of quadratic equations for ζ:

$$\zeta^2 - R(k) = \pm i\, 2\hat{k}c\Delta t\, \zeta \tag{10}$$

The four roots are

$$\begin{aligned}
\zeta_1 &= [R(k) - (\hat{k}c\Delta t)^2]^{\frac{1}{2}} + i\hat{k}c\Delta t \\
\zeta_2 &= [R(k) - (\hat{k}c\Delta t)^2]^{\frac{1}{2}} - i\hat{k}c\Delta t \\
\zeta_3 &= -[R(k) - (\hat{k}c\Delta t)^2]^{\frac{1}{2}} + i\hat{k}c\Delta t \\
\zeta_4 &= -[R(k) - (\hat{k}c\Delta t)^2]^{\frac{1}{2}} - i\hat{k}c\Delta t
\end{aligned} \tag{11}$$

We assume that $R(k) > 0$, and find that provided

$$\hat{k}c\Delta t < \sqrt{R(k)} \tag{12}$$

the solution will be in the form of a damped wave, i.e., it will be computationally stable. All roots have the same magnitude, viz.,

$$|\zeta| = \sqrt{R(k)} \tag{13}$$

In order to give the condition (12) more specificity, one may introduce (4) for $\hat{k}$ and get

$$\Delta t < \frac{\sqrt{2}\,\Delta x}{c}\,\frac{\sqrt{R(k)}}{(\sin k\Delta x)} \tag{14}$$

The $\sqrt{2}$ has been introduced to account for the general two-dimensional problem. The parameter c is not strictly constant and should be given a value appropriate for a wave moving in a "zonal" current. A reasonable estimate for c is 400 m/sec. The value assigned to Δx is 120 km, based upon the recognition that only north of 15° latitude are the equations of the LFM solved without interaction with the imposed boundary conditions. With these estimates inequality (14) may be rewritten as

$$\Delta t < \left(\frac{\sqrt{R(k)}}{\sin k\Delta x}\right) 424 \text{ sec.} \tag{15}$$

3. Computational Analysis and Experimental Results

When the RMB assumed responsibility for the LFM, the parameter β in the smoothing operator had the value 0.022, but the smoothing routine "SMOH" operated only upon the parameters, p_σ and θ, and only up to two grid intervals from the boundary. The Robert time filter was applied to the precipitable water, winds, p_σ and θ.

The first model modification introduced was the addition of the divergence damper term in the wind equations, but only in the region close to the boundary.

It was found that the sigma coordinate parameters retained considerable noise in spite of the smoothers employed in the model. In experiments, suggested by Mathur, we extended the smoothers up to the boundaries, varied the smoothing coefficient β and applied the smoother to the winds as well as p_σ and θ.

In November of this year, a modified smoother routine called SMOHEX was introduced into the operational model. The parameter β in the smoother was set at 0.030, and the smoother was applied as outlined to all variables except the precipitable water. The divergence damper and time filtering of all but precipitable water were deleted.

In spite of the improvement achieved with the SMOHEX routine and its minimal modification of the meteorological aspects of the model's forecasts, there remained certain problems of the type mentioned in the Introduction. The experiments designed to correct these deficiencies involved the variation of the parameter β in the SMOHEX routine. Results indicated that β values of .10 and .20 were both stable and also minimized the deficiencies of the forecasts. An experiment with $\beta = .25$ was unsuccessful due to an apparent computational instability.

In the experiments cited, the time step Δt was held at 360 secs, the presently operational value. Parallel experimentation was made with longer time steps, 400 and 450 secs,corresponding to nine and eight steps per hour versus the ten steps taken operationally in each forecast hour. The 400 sec time step was stable but the 450 sec step was not.

It has been found that these results are explicable by means of the mathematical analysis presented in section 2.

In figure 1, the response function R(k) is plotted for various choices of the parameter β. In figure 2, the maximum value of Δt satisfying the inequality (12) is plotted as a function of β and $k\Delta x$.

The instability of the computation with β equal to 0.25 and Δt equal to 360 secs is supported by the analysis. The maximum stable value of Δt is about 290 secs when $\beta = 0.25$. The choice of Δt equal to 360 secs is shown to be satisfactory for β equal to 0.20 or smaller values.

We also observe that with a zero value of β, i.e. no smoothing, linear stability is predicted for Δt less than 424 secs. This supports the stability observed in the experiments with small β (e.g., .022) and Δt equal to 400 secs and conversely the instability observed with Δt equal to 450 secs.

5. Diffusion and Smoothing Compared

The response function for β equal to 0.1 is almost unity for $k\Delta x$ less than $\pi/4$. The smoothing, however, repeatedly damps the solution as shown in equation (13). After 240 steps the response is given by

$$R_{240} = [\sqrt{R(k)}\,]^{240} \tag{16}$$

In figure 3, R_{240} is plotted for β equal to 0.1. One notes that for $k\Delta x$ equal to $\pi/4$ the value of R_{240} is only two-thirds. Ninety percent of the wave amplitude is retained only for those waves for which $k\Delta x$ is less than about $\pi/8$, which corresponds to a wave length of about sixteen grid intervals (say between two and four thousand kilometers).

Also shown in figure 3 are the response curves for a 24-hour integration using a Fickian type diffusion (evaluated at the n-1 time level) with constant diffusion coefficient between 5×10^4 and 5×10^5 $m^2 sec^{-1}$. Comparison of the curves shows, in the authors' opinion, some clear merits to the smoothing operator technique now used in the LFM.

The response for the diffusion operator is of the form

$$R_D(k) = + \frac{4\Delta t\, D}{(\Delta x)^2}(1 - \cos k\Delta x) \tag{17}$$

Comparison of this expression with that given in (8) shows that equivalence could be effected if the coefficient D was itself a function of k, i.e.,

$$D = \left(\frac{\beta^2(\Delta x)^2}{\Delta t}\right)(1 - \cos k\Delta x) \tag{18}$$

It should be noted that in some general circulation models, D is made a function of the wind field deformation. This will effect a variation of D with wave number since the deformation will tend to be larger as k increases. It is possible, therefore, that SMOHEX is simulating a non-linear diffusion.

6. Conclusion

The results obtained experimentally with the new LFM time integration technique have been shown to be explicable by means of a simple numerical analysis. Examination of the forecast charts indicates that the modifications are effective in selectively removing the spurious vorticity and sea level features while leaving the precipitation forecast almost unchanged.

It should also be noted in conclusion that the new technique has been integrated to 3 days, 72 hours, with very well behaved sigma and pressure coordinate fields.

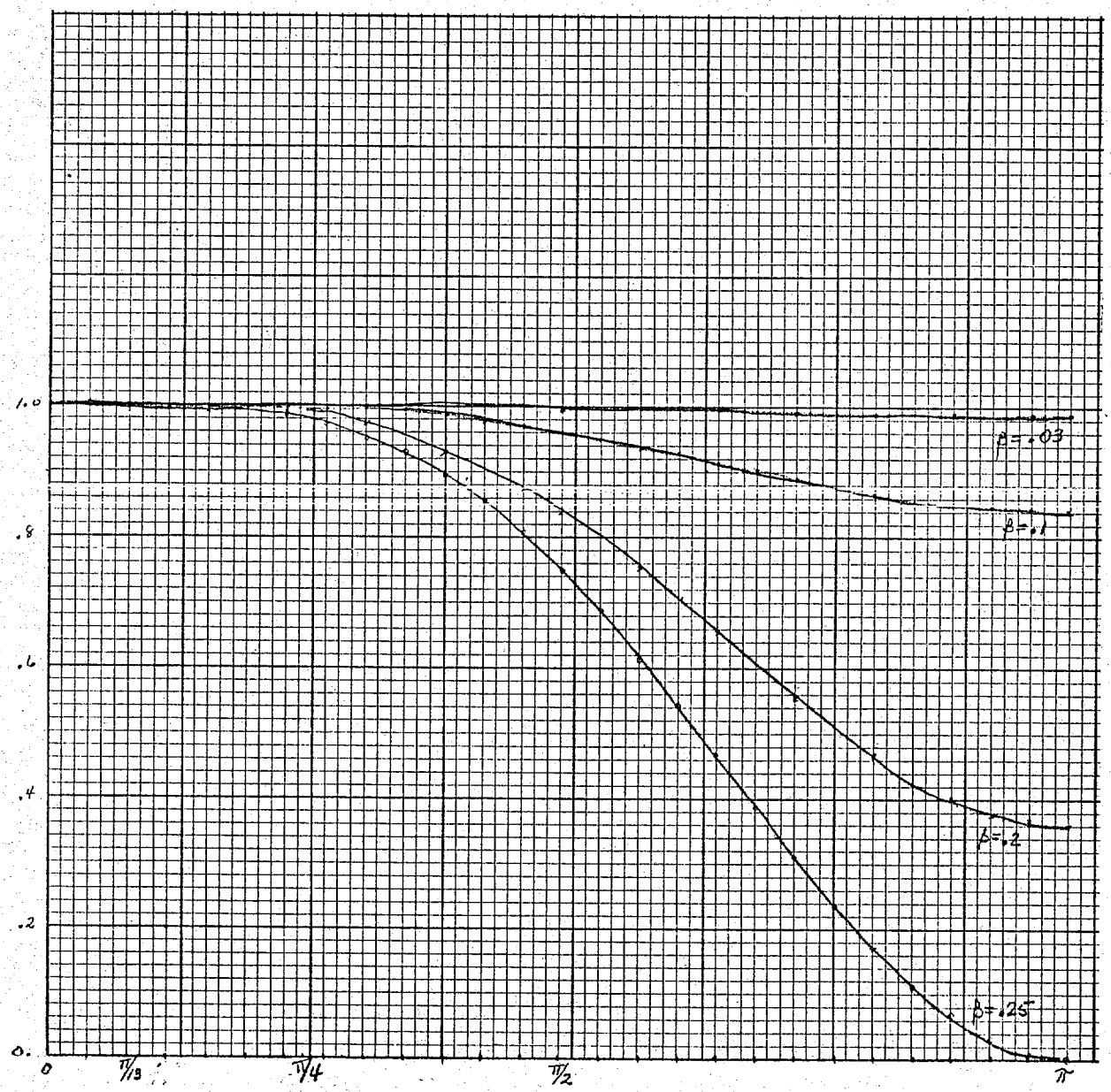


Figure (1)

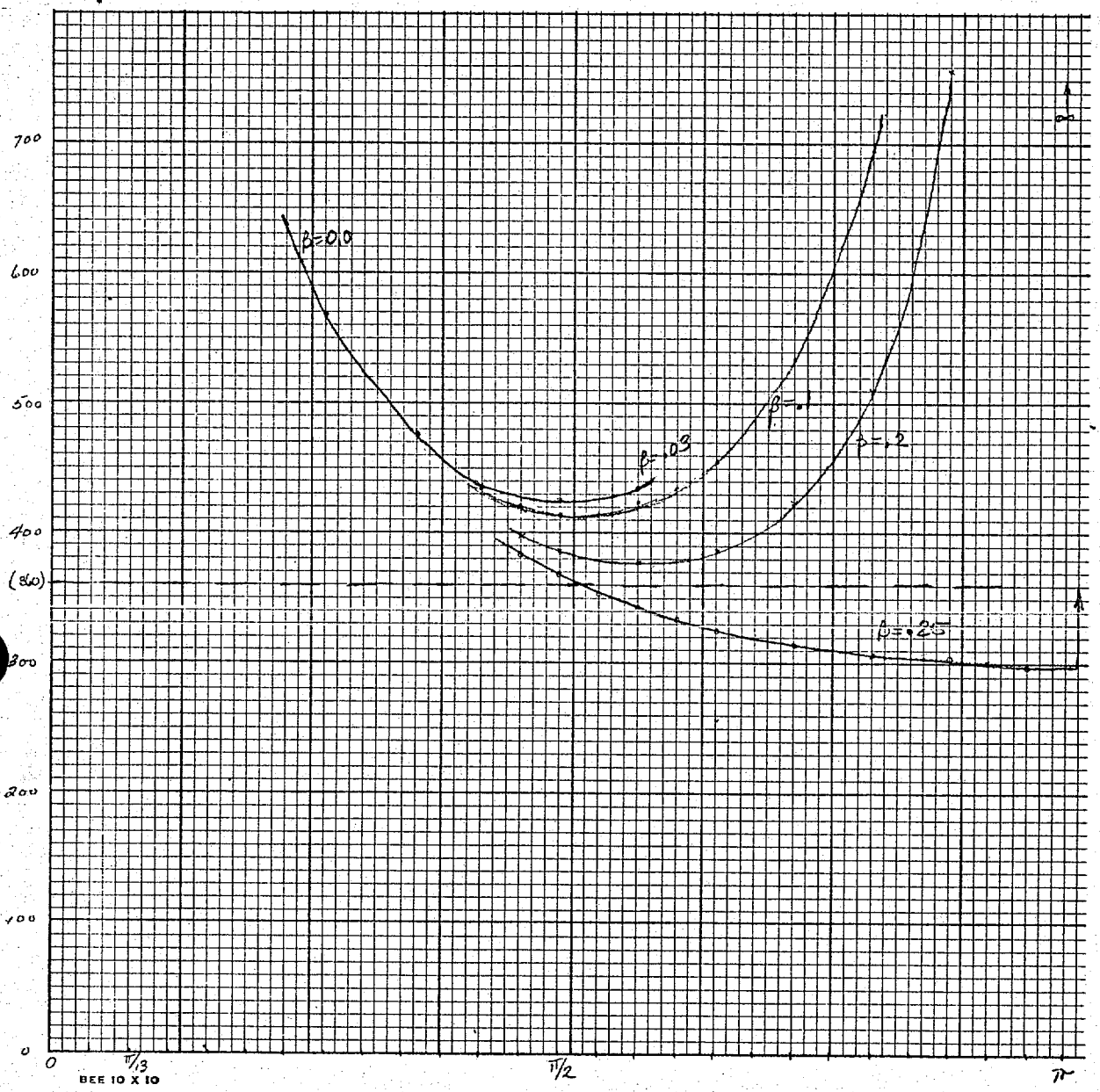
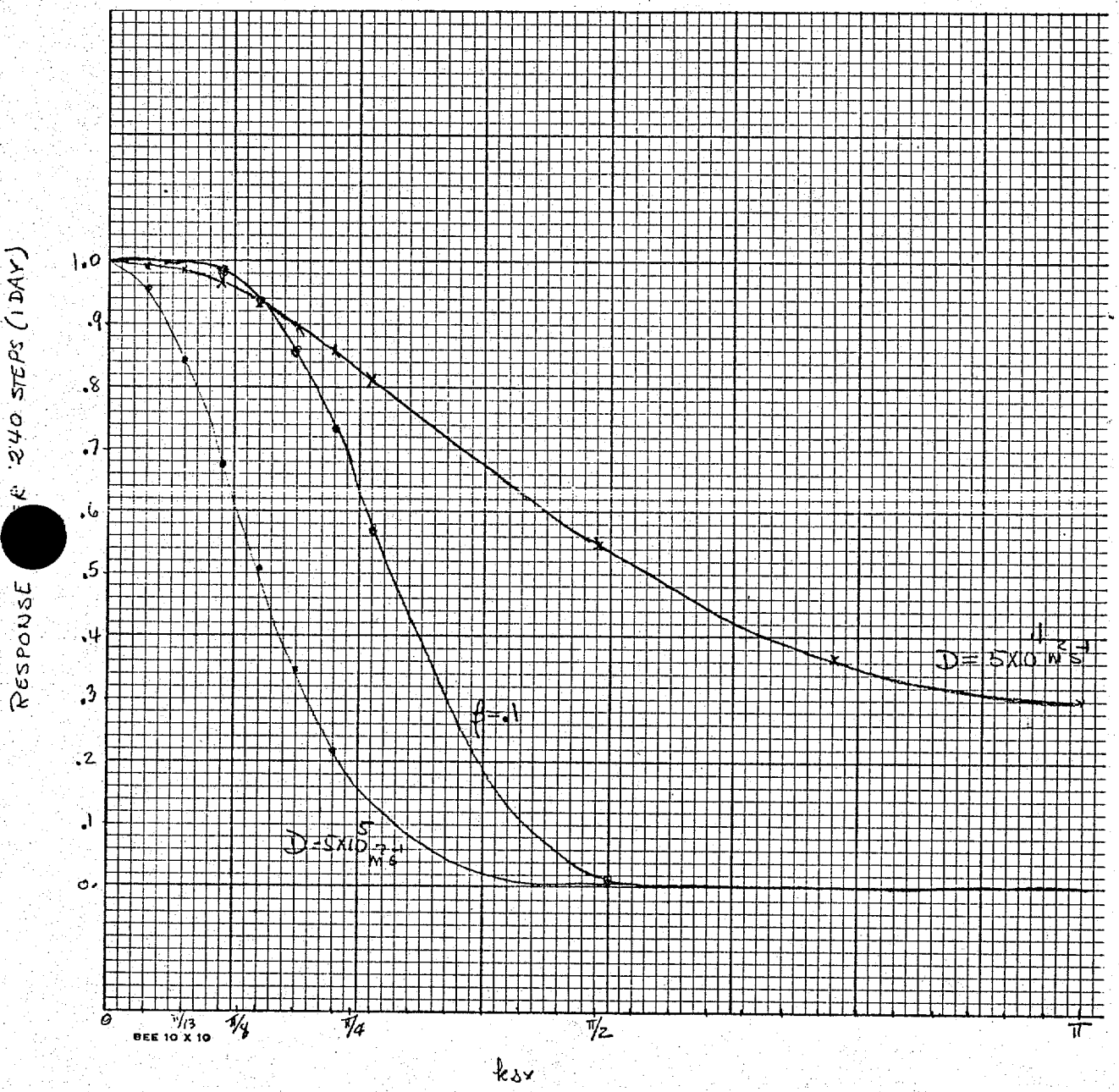


Figure 2.

$k_s x \rightarrow$



RESPONSE PER 240 STEPS (1 DAY)
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0.
0
π/13
π/8
π/4
π/2
π
BEE 10 X 10
ksx
β=.1
D=5X10^5 m^2 s^-1
D=5X10^4 m^2 s^-1

Figure 3.